The Parabolic Anderson Model on a Galton-Watson Tree

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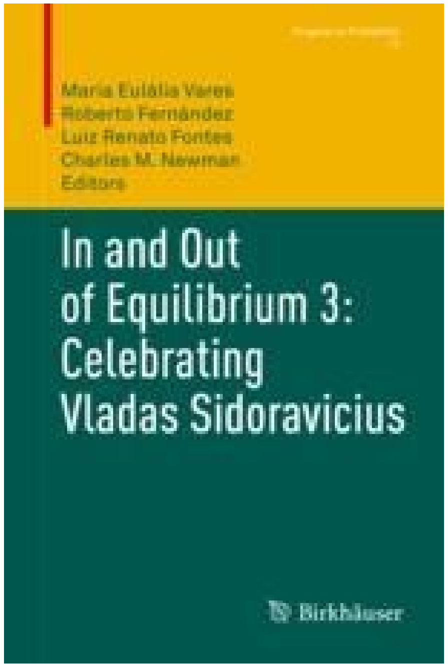
Vladas Sidoravicius
(1963-2019)

Vladas made seminal contributions to the area of random media. The topic of this lecture would have been very dear to his heart.

Vladas was an inspiring force in the mathematical physics community and is sorely missed by his colleagues. A tribute to his legacy is laid down in a beautiful memorial volume:



The work described is this lecture is joint with:


Wolfgang König (Berlin)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Daoyi Wang (Leiden)

- BACKGROUND

The Parabolic Anderson Model is the system of PDEs

$$\partial_t u(x,t) = (\Delta_{\mathcal{X}} u)(x,t) + \xi(x)u(x,t), \qquad x \in \mathcal{X},\ t > 0,$$

with $\mathcal{X}$ an ambient space, $\Delta_{\mathcal{X}}$ a Laplace operator acting on functions on $\mathcal{X}$, and ξ a random potential on $\mathcal{X}$.

Most of the literature considers the setting where $\mathcal{X} = \mathbb{Z}^d$ or $\mathcal{X} = \mathbb{R}^d$, and ξ is drawn from different probability laws.

Foundational papers: Gärtner, Molchanov 1990, 1998.

Many follow-up papers.

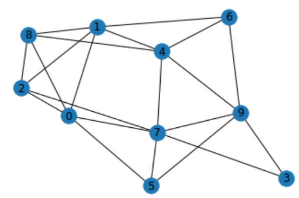
Monograph: König 2016.

More recently, other choices for $\mathcal{X}$ have been considered as well:

- deterministic graphs:
  complete graph, hypercube.

- random graphs:
  Galton-Watson tree, configuration model.

Much remains open for the latter category.



Literature:

Fleischmann, Molchanov 1990 + Avena, Gün, Hesse 2016.

dH, König, dos Santos 2020 + dH, Wang 2021 + work in progress.

For large t the solution of the PAM concentrates on well-separated regions in $\mathcal{X}$, called intermittent islands. Much of the literature focusses on a detailed description of the size, shape and location of these islands, and on the profiles of the potential and the solution on them.

A special role is played by the case where ξ is i.i.d. with a double-exponential marginal distribution

$$\mathrm{P}(\xi(0) > u) = \mathrm{e}^{-\mathrm{e}^{u/\varrho}}, \qquad u \in \mathbb{R},$$

where $\varrho \in (0, \infty)$ is a parameter. This distribution turns out to be critical, in the sense that the intermittent islands neither grow nor shrink with time, and therefore represents a class of its own.

In the present lecture we focus on the case where $\mathcal{X}$ is a Galton-Watson tree, and consider two settings:

- Quenched:
  almost surely with respect to the random tree and the random potential.

- Half-annealed:
  almost surely with respect to the random tree, but averaged over the random potential.

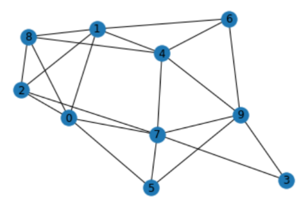
It will turn out that the behaviour of the PAM is different in these two settings.

- THE PAM ON A GRAPH

Let $G = (V, E)$ be a simple connected undirected graph, either finite or countably infinite, with a designated vertex $\mathcal{O}$ called the root. Let Δ_G be the Laplacian on G, i.e.,

$$(\Delta_G f)(x) = \sum_{\substack{y \in V: \\ \{x,y\} \in E}} [f(y) - f(x)], \qquad x \in V,\ f\colon V \to \mathbb{R},$$

which acts along the edges of G. Let $\xi = (\xi(x))_{x \in V}$ be a random potential attached to the vertices of G, taking values in $\mathbb{R}$.



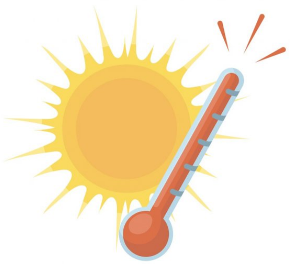
Our object of interest is the PAM with a localised initial condition:

$$\begin{aligned} \partial_t u(x,t) &= (\Delta_G u)(x,t) + \xi(x)u(x,t), & x \in V,\ t > 0, \\ u(x,0) &= \delta_{\mathcal{O}}(x), & x \in V. \end{aligned}$$

$u(x,t)$ can be interpreted as the amount of heat at time t at vertex x, when initially there is unit heat at $\mathcal{O}$ and $\xi(x)$ acts as a source or sink.

The total heat at time t is

$$U(t) = \sum_{x \in V} u(x,t).$$



▶ FEYNMAN-KAC REPRESENTATION

The quenched total heat at time t can be represented by the Feynman-Kac formula

$$U(t) = \mathbb{E}_{\mathcal{O}}\left(\mathrm{e}^{\int_0^t \xi(X_s)\mathrm{d}s}\right),$$

where $X = (X_t)_{t\geq 0}$ is the continuous-time random walk on the vertices V with jump rate 1 along the edges E, and $\mathbb{P}_{\mathcal{O}}$ denotes the law of X given $X_0 = \mathcal{O}$.

Note that three types of randomness are in play: random tree, random potential, random walk.

Let $\langle \cdot \rangle$ denote expectation with respect to ξ. The annealed total heat at time t is

$$\langle U(t) \rangle = \left\langle \mathbb{E}_{\mathcal{O}} \left(\mathrm{e}^{\int_0^t \xi(X_s) \mathrm{d}s} \right) \right\rangle .$$

If we assume that the random potential $\xi = (\xi(x))_{x \in V}$ is i.i.d. with marginal cumulant generating function

$$H(u) = \log \left\langle \mathrm{e}^{u\xi(\mathcal{O})} \right\rangle , \qquad u \geq 0,$$

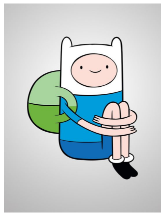
then

$$\langle U(t) \rangle = \mathbb{E}_{\mathcal{O}} \left(\exp \left[\sum_{x \in V} H(\ell_t^X(x)) \right] \right) ,$$

where

$$\ell_t^X(x) := \int_0^t 1_{\{X_s = x\}} \, \mathrm{d}s, \quad x \in V, \, t \geq 0,$$

is the local time of X at vertex x up to time t.



▶ KEY VARIATIONAL FORMULA

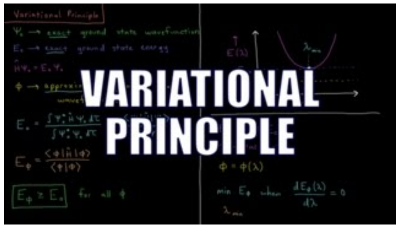
Denote by $\mathcal{P}(V)$ the set of probability measures on V. For $p \in \mathcal{P}(V)$, define

$$I_E(p) = \sum_{\{x,y\}\in E} \left(\sqrt{p(x)} - \sqrt{p(y)}\right)^2,$$

$$J_V(p) = -\sum_{x\in V} p(x) \log p(x),$$

and set

$$\chi_G(\varrho) = \inf_{p\in\mathcal{P}(V)} [I_E(p) + \varrho J_V(p)], \qquad \varrho \in (0,\infty).$$



The first term is the quadratic form associated with the Laplacian, which is the large deviation rate function for the empirical distribution

$$L_t^X = \frac{1}{t}\int_0^t \delta_{X_s}\,\mathrm{d}s = \frac{1}{t}\sum_{x\in V}\ell_t^X(x)\delta_x.$$

Donsker,Varadhan 1975

The second term captures the asymptotics of the cumulant generating function H.

▶ THE PAM ON A GALTON-WATSON TREE

Denote by $\mathcal{GW} = (V, E)$ the Galton-Watson tree with root $\mathcal{O}$ and offspring distribution D. Write $\mathfrak{P}$ to denote its law. Suppose that

$$d_{\mathsf{min}} = \min \mathrm{supp}(D) \geq 2, \qquad \mathrm{mean}(D) \in (2, \infty).$$

Under this assumption, $\mathcal{GW}$ is $\mathfrak{P}$-a.s. an infinite tree, and

$$\lim_{R \to \infty} \frac{\log |B_R(\mathcal{O})|}{R} = \log \mathrm{mean}(D) = \vartheta \in (0, \infty) \qquad \mathfrak{P} - a.s.,$$

where $B_R(\mathcal{O}) \subset V$ is the ball of radius R around $\mathcal{O}$ in the graph distance.

Suppose that

$$\lim_{u\to\infty} uH''(u) = \varrho \in (0,\infty),$$

which is in fact a neighbourhood of the double-exponential distribution. Write $\chi(\varrho)$ to denote the variational formula with $G = \mathcal{GW}$.

THEOREM quenched growth rate

Suppose that $\text{mean}(\mathrm{e}^{\mathrm{e}^{\mathrm{e}^{aD}}}) < \infty$ *for some* $a > 0$*. Then*

$$\frac{1}{t}\log U(t) = \varrho \log\left(\frac{\varrho t \vartheta}{\log\log t}\right) - \varrho - \chi(\varrho) + o(1) \quad \mathrm{P}\times\mathfrak{P}\text{-a.s.}$$

THEOREM half-annealed growth rate

Suppose that $\text{mean}(\mathrm{e}^{aD}) < \infty$ *for some* $a > \vartheta$*. Then*

$$\frac{1}{t}\log\langle U(t)\rangle = \varrho\log(\varrho t) - \varrho - \chi(\varrho) + o(1) \quad \mathfrak{P}\text{-a.s.}$$

▶ DISCUSSION

1. It can be shown that

$$\chi(\varrho) = \inf_{\mathcal{T}} \chi_{\mathcal{T}}(\varrho),$$

where the infimum runs over all infinite trees with degrees in $\mathrm{supp}(D)$. In other words, the variational formula on $\mathcal{GW}$ fully concentrates on an optimal tree contained in $\mathcal{GW}$.

It can be shown that if $\varrho \geq 1/\log(d_{\min}+1)$, then the unique optimal tree is $\mathcal{T}_{d_{\min}}$, the regular tree with degree $d_{\min}+1$. Possibly this is the optimal tree for all $\varrho > 0$.

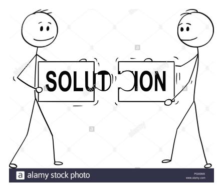
2. The quenched asymptotics requires more stringent conditions on the tail of the offspring distribution D than the half-annealed asymptotics.

The extra term in the quenched asymptotics comes from the cost for X to

- travel in a time of order $o(t)$ to an optimal tree with an optimal profile of the potential, located at a distance of order $\varrho t/\log\log t$ from $\mathcal{O}$,
- subsequently spend most of its time on that tree.

In this cost, the parameter ϑ appears, which is absent in the half-annealed asymptotics.

3. The proof of the two theorems is obtained by deriving asymptotically matching upper and lower bounds. These are obtained by truncating $\mathcal{GW}$ after generation R, deriving the asymptotics on $\mathcal{GW}_R$ for finite R, and letting $R \to \infty$ afterwards.



• For the lower bound we can use the standard truncation technique, which is based on killing X when it exits $\mathcal{GW}_R$ and applying the large deviation principle for the empirical distribution of Markov processes.

• For the upper bound the standard truncation technique is based on periodisation of X on $\mathcal{GW}_R$, which fails because $\mathcal{GW}$ is a random expander graph. Instead, we use projection of X on $\mathcal{GW}_R$ and apply the large deviation principle for the empirical distribution of Markov renewal processes.

Mariani, Zambotti 2016.

I wish Vladas was here to smile and to ask questions!