Generalized Excited Random Walk in Bernoulli Environment

Probability Seminar IM-UFRJ

Rodrigo Barreto Alves (FGV-EMAp)

Joint work with Giulio Iacobelli (IM-UFRJ) e Glauco Valle (IM-UFRJ)

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Outline

Excited random walk (ERW) is a model introduced by Benjamini and Wilson '03. Also called Cookie Random Walk.

It's a discrete time RW in $\mathbb{Z}^d$, $d \geq 2$ (starting from the origin).

- It depends on a fixed parameter $\delta \in (1/2, 1]$.
- At the first visit to a site, it will jump in the following way:

$$p(x, e_1) = \delta/d, \quad p(x, -e_1) = (1-\delta)/d$$

  and $\forall i \in \{2, 3, \dots, d\}$

$$p(x, \pm e_i) = 1/2d$$

- On an already visited site, the RW jumps to any nearest neighbor with uniform probability.

Introduction

Benjamini and Wilson '03 proved that ERW in $\mathbb{Z}^d$, $d \geq 2$ is transient to the right

$$\lim_{n\to\infty} X_n \cdot e_1 = \infty \qquad \text{a.s..}$$

Furthermore, they also show that, if $d \geq 4$, ERW is ballistic to the right

$$\liminf_{n\to\infty} \frac{X_n \cdot e_1}{n} > 0 \qquad \text{a.s..}$$

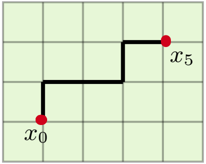
Subsequent results:

- Kozma '03 and '05 extended the proof of ballisticity for ERW to $d = 3$ and $d = 2$, respectively.
- Bérnard and Ramirez '07 proved a Law of Large Numbers and a Central Limit Theorem for ERW in dimension $d \geq 2$.

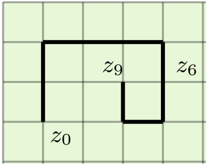
The proofs of the transience to the right, the Law of Large Numbers and the Central Limit Theorem, rest upon two important ingredients.

- Couple the ERW with the SRW.
- *Tan points* for the SRW.

A tan point in $d = 2$ is a site $x \in \mathbb{Z}^2$ such that x is the first site of $\{x + ke_1; k \geq 0\}$ visited by the SRW.



x_5 *is a tan point*



z_9 *is not a tan point*

A more robust technique was developed by Menshikov, Popov, Ramirez and Vachkovskaia '12. The model is a discrite time process in $\mathbb{Z}^d$, $d \geq 2$ and they considered the following:

- on already visited sites the process behaves like a d-dimensional martingale with bounded jumps (rather than a SRW),
- on the first time a site is visited the process has bounded jumps, satisfies UEC and drift condition in an arbitrary direction ℓ.

They call this model *generalized excited random walk* (GERW) and they showed that GERW with a drift condition in direction ℓ, is ballistic in that direction.

The Model

Let $X = \{X_n\}_{n \geq 0}$ be a $\mathbb{Z}^d$ valued process, with $d \geq 2$, $X_0 = 0$ and adapted to a filtration $\mathcal{F} = \{\mathcal{F}_n\}_{n \geq 0}$.

<u>Condition I</u>: There exists a constant $K > 0$ such that $\sup_{n \geq 0} ||X_{n+1} - X_n|| < K$ on every realization.

We have that $\pi = \{\pi(x)\}_{x \in \mathbb{Z}^d} \in [0,1]^{\mathbb{Z}^d}$ is a random element where the marginals are Uniform in $[0,1]$ and independents. We fix a sequence $\{p_n\}_{n \geq 1}$, with $p_n \in (0,1]$. Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n, \pi(X_1), \ldots, \pi(X_n))$ and $\ell \in \mathbb{S}^{d-1}$.

The Model

Condition II: There exists a $\lambda > 0$ such that:

- on the event $\{X_k \neq X_n \text{ , for all } k < n\}$,

$$\begin{aligned} &\mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] \cdot \ell \geq \lambda \,, && \text{if } \pi(X_n) \leq p_n \,, \\ &\mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] = 0 \,, && \text{if } \pi(X_n) > p_n \,. \end{aligned}$$

- on the event $\{\exists\, k < n \text{ such that } X_k = X_n\}$,

$$\mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] = 0 \,.$$

Condition III: There exist $h, r > 0$ such that;

- *Uniformly elliptic in direction ℓ:* for all n,

$$\mathbb{P}\left[(X_{n+1} - X_n) \cdot \ell > r | \mathcal{F}_n\right] \geq h \ , \tag{UE1}$$

- *Uniformly elliptic on the event $\{\mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] = 0\}$:* on $\{\mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] = 0\}$, for all $\ell' \in \mathbb{S}^{d-1}$, with $||\ell'|| = 1$,

$$\mathbb{P}\left[(X_{n+1} - X_n) \cdot \ell' > r | \mathcal{F}_n\right] \geq h \ . \tag{UE2}$$

We will call X a p_n-GERW.

Main Result for the p_n-GERW

Let $\{\eta(X_0) = \infty\}$ be the event in which the process X never returns to the origin in the drift direction.

Theorem

Let X be a p_n-GERW in direction ℓ, in $\mathbb{Z}^d$ with $d \geq 2$, where $p_n = (q_0 + n)^{-\beta}$, with $\beta < 1/6$, q_0 is a non negative integer. There exists $\psi > 0$ depending on the parameters of the model such that

$$\mathbb{P}\left[\eta(X_0) = \infty\right] \geq \mathbb{P}\left[X_n \cdot \ell > 0 \text{ for all } n \geq 1\right] \geq \psi\,.$$

Writing p_n-GERW

Let $\{X_n\}_{n\geq 0}$ be a p_n-GERW in direction ℓ and $\{U_i\}_{i\geq 1}$ a sequence of i.i.d. random variables with uniform distribution in $[0,1]$. Denote the event $E_i := \{\exists\, k < i \text{ such that } X_k = X_i\}$ and $E_0 := \emptyset$.

$$\begin{aligned} X_n &= \sum_{i=1}^{n}(X_i - X_{i-1}) \\ &= \sum_{i=1}^{n}\left(1_{\{E_{i-1}\}}\xi_i + 1_{\{E_{i-1}^c\}}1_{\{U_i > p_i\}}\xi_i + 1_{\{E_{i-1}^c\}}1_{\{U_i\leq p_i\}}\gamma_i\right). \\ &= \sum_{i=1}^{n}\left(\xi_i + 1_{\{E_{i-1}^c\cap\{U_i\leq p_i\}\}}\gamma_i - 1_{\{E_{i-1}^c\cap\{U_i\leq p_i\}\}}\xi_i\right). \end{aligned}$$

We set $\{\xi_i, \mathcal{F}_i\}_{i\geq 1}$ is an increment of a d-martingale with zero mean and $\{\gamma_i, \mathcal{F}_i\}_{i\geq 1}$ is a random vector such that $\mathbb{E}[\gamma_i \cdot \ell | \mathcal{F}_{i-1}] \geq \lambda$ for all $i \geq 1$.

We set a polynomial decay: $p_n = \mathcal{C}n^{-\beta} \wedge 1$ with $\beta > 0$ and $\mathcal{C} > 0$.

A weaker version of the p_n-GERW (p_n-WGERW)

Denote $C = ((c_{i,j}))$ a continuous, $d \times d$ matrix-valued real function, defined in $[0, \infty)$, satisfying $C(0) = 0$ and $\sum_{i,j=1}^{d}(c_{i,j}(t) - c_{i,j}(s))\alpha_i\alpha_j \geq 0$ for any $\alpha \in \mathbb{R}^d$, $t > s \geq 0$.

Condition I*:

i) For all $k \geq 1$ and $\theta < \beta - 1/2$, where $\beta > 1/2$, we have

$$\sup_{k\geq 1} \frac{\mathbb{E}[\|\gamma_k\|]}{k^\theta} < \infty \quad \text{and} \quad \sup_{k\geq 1} \frac{\mathbb{E}[\|\xi_k\|]}{k^\theta} < \infty\,.$$

ii) When the process behaves like $\{\xi_i\}_{i\geq 0}$

$$\frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor} \xi_i \xi_i^T \to C(t) \quad \text{as } n \to \infty\,,$$

in probability and

$$\lim_{k\to\infty} k^{-1/2}\mathbb{E}\left[\sup_{1\leq i\leq k} \|\xi_i\|\right] = 0\,.$$

Main Result for the p_n-WGERW

We define the following process

$$\hat{B}_t^n = \frac{X_{\lfloor nt \rfloor}}{n^{1/2}} + (nt - \lfloor nt \rfloor)\frac{(X_{\lfloor nt \rfloor + 1} - X_{\lfloor nt \rfloor})}{n^{1/2}}.$$

$C_{\mathbb{R}^d}[0,T]$ with uniform metric and $C_{\mathbb{R}^d}[0,\infty)$ with the metric $\rho(f,g) := \sum_{k=1}^{\infty} \frac{1}{2^k} \sup_{0 \le t \le k}(||f(t) - g(t)|| \wedge 1)$.

Theorem

Let X be a p_n-WGERW in direction ℓ, in $\mathbb{Z}^d$, with $d \geq 2$, $p_n = \mathcal{C}n^{-\beta} \wedge 1$, with $\beta > 1/2$. Then $\hat{B}^n_{\cdot}$ converges in distribution to a unique, in distribution, process with independent Gaussian increments with sample paths in $C_{\mathbb{R}^d}[0,\infty)$.

A specific case of p_n-GERW (p_n-ERW)

Let X be a p_n-GERW in direction ℓ

$$X_n = \sum_{i=1}^{n} \left(\xi_i + 1_{\{E_{i-1}^c \cap \{U_i \leq p_i\}\}} \gamma_i - 1_{\{E_{i-1}^c \cap \{U_i \leq p_i\}\}} \xi_i \right).$$

If the sequence $\{\xi_i\}_{i \geq 1}$ is i.i.d. with zero mean and finite variance and $\{\gamma_i\}_{i \geq 1}$ is also i.i.d. with finite variance, then X is p_n-ERW in the direction ℓ.

The Range of p_n-ERW in $d \geq 2$ and $\beta = 1/2$

Given a process $\{X_n\}_{n\geq 0}$ on the lattice $\mathbb{Z}^d$, we denote its *range* at time n by,

$$\mathcal{R}_n^X = \{x \in \mathbb{Z}^d : X_k = x \text{ for some } 0 \leq k \leq n\} .$$

Let π_d be the probability of a random walk with i.i.d. increments never returning to the origin.

Proposition

Let X be a p_n-ERW in direction ℓ, in $\mathbb{Z}^d$ with $d \geq 2$, $p_n = Cn^{-1/2} \wedge 1$. Then, we have that

$$\mathbb{P}\left[|\mathcal{R}_n^X| \leq \delta n\right] \to 1 \quad \text{as } n \to \infty ,$$

for every $\delta > \pi_d$ corresponding to $\{\xi_i\}_{i\geq 0}$.

Main Result for the p_n-ERW

Theorem

Let X be a p_n-ERW in direction ℓ, in $\mathbb{Z}^d$ with $d = 2$, $p_n = Cn^{-1/2} \wedge 1$. Then $\hat{B}^n_\cdot$ converges in distribution to a 2-dimensional Brownian Motion in $C_{\mathbb{R}^2}[0, \infty)$.

Main Result for the p_n-ERW

Define the set $D_k \subset \{e_1, \ldots, e_d\}$, where $d \geq 4$ and $|D_k| = k$, with $1 \leq k \leq d-3$.

Let $\ell_{D_k} \in \mathbb{S}^{d-1}$, such that $\ell_{D_k} = \sum_{i=1}^{k} \alpha_i x_i$, where $\alpha_i \in [0,1]$ and $x_i \in D_k$, both for all $1 \leq i \leq k$.

Theorem

Let X be a p_n-ERW in direction ℓ_{D_k}, in $\mathbb{Z}^d$ with $d \geq 4$, $p_n = Cn^{-1/2} \wedge 1$. Then the process $\hat{B}^n_\cdot$ is tight in $C_{\mathbb{R}^d}[0,\infty)$ and there exists a Brownian Motion $W_\cdot$ such that for every limit point $Y_\cdot$ of the process $\hat{B}^n_\cdot$ it holds that

$$W_t \cdot \ell_{D_k} + c_1\sqrt{t} \preceq Y_t \cdot \ell_{D_k} \preceq W_t \cdot \ell_{D_k} + c_2\sqrt{t}\,,$$

where $c_2 > c_1 > 0$.

Meaning of our result for the p_n-ERW

Every limit point of the p_n-ERW in direction ℓ_{D_k} suitably rescaled will be in a kind a "cone" region, with high probability.

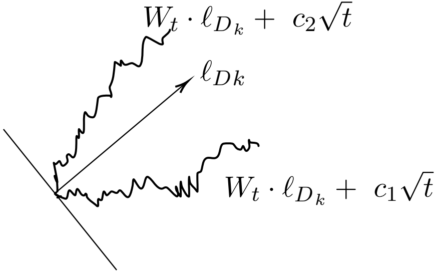


Figure 1: "Cone" region representation around the direction ℓ_{D_k}.

Summary of Results

p_n-GERW $(\beta < 1/6,\ d \geq 2)$	positive probability of never returning to the origin (in the direction ℓ)
p_n-WGERW $(\beta > 1/2,\ d \geq 2)$	convergence in distribution to a Gaussian Process
p_n-ERW $(\beta = 1/2,\ d = 2)$	convergence in distribution to a Brownian Motion
p_n-ERW $(\beta = 1/2,\ d \geq 4)$	all sub-sequences converge, in distribution, to a process which is stochastically dominated in the drift direction below and above by a Brownian Motion plus a continuous function.

Idea of Proof of the convergence in distribution of the p_n-WGERW

For simplicity

$$B^n_{\cdot} := \frac{X_{\lfloor n\cdot \rfloor}}{n^{1/2}}\,.$$

Let X be a p_n-WGERW in direction ℓ, $d \geq 2$ and with $\beta > 1/2$. Then

$$B^n_{\cdot} \xrightarrow{\mathcal{D}} Z_{\cdot} \quad \text{as } n \to \infty\,,$$

where Z is a unique, in distribution, process with independent Gaussian increments.

Idea of Proof of the convergence in distribution of the p_n-WGERW

- We write the process

$$B_t^n = \frac{1}{n^{1/2}} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i + \frac{1}{n^{1/2}} \sum_{i=1}^{\lfloor nt \rfloor} 1_{\{E_{i-1}^c \cap \{U_i \le i^{-\beta}\}\}} (\gamma_i - \xi_i)\,.$$

- By Condition I* we have $n^{-1} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i \xi_i^T \xrightarrow{\mathcal{P}} C(t)$ as $n \to \infty$ and $k^{-\frac{1}{2}} \mathbb{E}\left[\sup_{1 \le i \le k} \|\xi_i\|\right] \to 0$ as $k \to \infty$.

- Then by Theorem in Ethier and Kurtz '09 we obtain

$$\frac{1}{n^{1/2}} \sum_{i=1}^{\lfloor n \cdot \rfloor} \xi_i \xrightarrow{\mathcal{D}} Z. \quad \text{as } n \to \infty\,.$$

Idea of Proof of the convergence in distribution of the p_n-WGERW

- We define $D^{\gamma}_{\lfloor n\cdot \rfloor} := \frac{1}{n^{1/2}} \sum_{i=1}^{\lfloor nt \rfloor} 1_{\{E^c_{i-1}\}} 1_{\{U_i \leq i^{-\beta}\}} \gamma_i$, a process in $C_{\mathbb{R}^d}[0,T]$
- We have

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} \left\| D^{\gamma}_{\lfloor nt \rfloor} \right\| > \varepsilon \right) \leq \mathbb{P}\left(\sum_{i=1}^{\lfloor nT \rfloor} \left\| 1_{\{U_i \leq i^{-\beta}\}} \gamma_i \right\| > \varepsilon n^{\frac{1}{2}} \right)$$

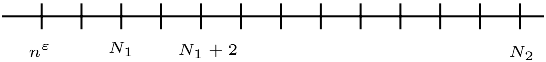
$$\leq \frac{1}{n^{1/2}\varepsilon} \sum_{i=1}^{\lfloor nT \rfloor} \frac{1}{i^{\beta}} \mathbb{E}\left[\|\gamma_i\| \right] \leq \frac{1}{n^{1/2}\varepsilon} \sum_{i=1}^{\lfloor nT \rfloor} \frac{\mathbb{E}\left[\|\gamma_i\| \right]}{i^{\theta}} \times \frac{1}{i^{\beta-\theta}}.$$

- By Condition I* we obtain that $D^{\gamma}_{\lfloor nt \rfloor} \xrightarrow{\mathcal{P}} 0$, in the space $C_{\mathbb{R}^d}[0,T]$ for all $T > 0$.

Idea of the proof of the upper bound to the range of p_n-ERW in $d \geq 2$ and $\beta = 1/2$

We want to prove that $\mathbb{P}[|\mathcal{R}_n^X| < \delta n] \to 1$ as $n \to \infty$ for every $\delta > \pi_d$ corresponding to $\{\xi_i\}_{i \geq 0}$.

- For an $\varepsilon \in (0,1)$ we have



- We think in each time window like $[N_1 + 2, N_2]$ has a independent random walk Y with i.i.d. increments. Each one with its range in this time window.

- Then we use the ranges of these processes to upper bound the range of the p_n-ERW.

Idea of the proof of convergence in distribution of the p_n-ERW, $d = 2$, $\beta = 1/2$

For simplicity

$$B^n_\cdot := \frac{X_{\lfloor n\cdot \rfloor}}{n^{1/2}}\,.$$

Let X be a p_n-ERW in direction ℓ, $d = 2$ and with $\beta = 1/2$. Then

$$B^n_\cdot \xrightarrow{\mathcal{D}} W_\cdot \quad \text{as } n \to \infty\,,$$

where W is a Brownian Motion.

Idea of the proof of convergence in distribution of the p_n-ERW, $d = 2$, $\beta = 1/2$

- We write the process

$$B_t^n = \frac{1}{n^{1/2}} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i + \frac{1}{n^{1/2}} \sum_{i=1}^{\lfloor nt \rfloor} 1_{\{E_{i-1}^c \cap \{U_i \leq i^{-1/2}\}\}} (\gamma_i - \xi_i)\,.$$

- By Donsker's Theorem $\frac{1}{n^{1/2}} \sum_{i=1}^{\lfloor n\cdot \rfloor} \xi_i \xrightarrow{\mathcal{D}} W.$ as $n \to \infty$, where W is a Brownian Motion with zero mean vector and covariance matrix $\mathbb{E}[\xi_1 \xi_1^T]$.
- Denote the set $K_n := \{i \in \{1, 2, \dots, n\} : 1_{\{E_{i-1}^c\}} 1_{\{U_i \leq i^{-1/2}\}} = 1\}$ and the sequence $\mathcal{F}$-stopping times $\{\tau_i\}_{i \geq 1}$, corresponding to the times the p_n-ERW visits a new site.

Idea of the proof of convergence in distribution of the p_n-ERW, $d = 2$, $\beta = 1/2$

We rewrite the process

$$B_t^n = \frac{1}{n^{1/2}} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i + \frac{|K_{\lfloor nt \rfloor}|}{n^{1/2}} \sum_{i \in K_{\lfloor nt \rfloor}} \frac{(\gamma_i - \xi_i)}{|K_{\lfloor nt \rfloor}|} .$$

By the definition of $K_{\lfloor nt \rfloor}$ we have

$$|K_{\lfloor nt \rfloor}| = \sum_{i=1}^{\lfloor nt \rfloor} 1_{\{E_{i-1}^c\}} 1_{\{U_i \leq i^{-1/2}\}} = \sum_{j=1}^{|\mathcal{R}_{\lfloor nt \rfloor}^X|} 1_{\{U_{\tau_j} \leq \tau_j^{-1/2}\}} \preceq \underbrace{\sum_{i=1}^{|\mathcal{R}_{\lfloor nt \rfloor}^X|} 1_{\{U_i \leq i^{-1/2}\}}}_{|J_{\lfloor nt \rfloor}| :=} .$$

Lemma

We have that the process $|J_{\lfloor n \cdot \rfloor}|/n^{1/2}$ converges in $C_{\mathbb{R}}[0, \infty)$ to the identically zero function in probability.

Idea of the proof of convergence in distribution of the p_n-ERW, $d = 2$, $\beta = 1/2$

- For any $\varepsilon > 0$, we set the event $G := \{|J_{\lfloor nT \rfloor}| > \varepsilon\sqrt{n}\}$. Then for a $\delta > 0$, we have that

$$\mathbb{P}\left[\sup_{0\leq t\leq T} |J_{\lfloor nt \rfloor}| > \varepsilon\sqrt{n}\right] =$$
$$= \mathbb{P}[G \cap \{|\mathcal{R}^X_{\lfloor nT \rfloor}| > \delta\lfloor nT \rfloor\}] + \mathbb{P}[G \cap \{|\mathcal{R}^X_{\lfloor nT \rfloor}| \leq \delta\lfloor nT \rfloor\}]$$

- Since $|K_{\lfloor n\cdot \rfloor}| \preceq |J_{\lfloor n\cdot \rfloor}|$, we obtain $|K_{\lfloor n\cdot \rfloor}|/n^{1/2} \xrightarrow{\mathcal{P}} 0$ as $n \to \infty$ in the space $C_{\mathbb{R}}[0, \infty)$.
- As $n \to \infty$ either $|K_{\lfloor nt \rfloor}| < \infty$ a.s. or $|K_{\lfloor nt \rfloor}| = \infty$ a.s..

Idea of the proof of convergence in distribution of the p_n-ERW, $d \geq 4$, $\beta = 1/2$

For simplicity

$$B^n_\cdot := \frac{X_{\lfloor n \cdot \rfloor}}{n^{1/2}} \,.$$

Let X be a p_n-ERW in direction ℓ_{D_k}, $d \geq 4$ and with $\beta = 1/2$. Then

$$W_t \cdot \ell_{D_k} + c_1 \sqrt{t} \preceq Y_t \cdot \ell_{D_k} \preceq W_t \cdot \ell_{D_k} + c_2 \sqrt{t} \,,$$

where $c_2 > c_1 > 0$.

Idea of the proof of convergence in distribution of the p_n-ERW, $d \geq 4$, $\beta = 1/2$

- We write the process

$$B_t^n = \frac{1}{n^{1/2}} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i + \frac{1}{n^{1/2}} \sum_{i=1}^{\lfloor nt \rfloor} 1_{\{E_{i-1}^c \cap \{U_i \leq i^{-1/2}\}\}} (\gamma_i - \xi_i)$$

- Let us set $D_{\lfloor nt \rfloor} := \frac{1}{n^{1/2}} \sum_{i=1}^{\lfloor nt \rfloor} 1_{\{E_{i-1}^c \cap \{U_i \leq i^{-1/2}\}\}} (\gamma_i - \xi_i)$. The process $D_{\lfloor n \cdot \rfloor}$ is tight in $C_{\mathbb{R}^d}[0, \infty)$.
- Note that

$$\sum_{i=n-|\mathcal{R}_{\lfloor nt \rfloor}^X|+1}^{\lfloor nt \rfloor} 1_{\{U_i \leq i^{-1/2}\}} \preceq \sum_{j=1}^{|\mathcal{R}_{\lfloor nt \rfloor}^X|} 1_{\{U_{\tau_j} \leq \tau_j^{-1/2}\}} = |K_{\lfloor nt \rfloor}|$$

Idea of the proof of convergence in distribution of the p_n-ERW, $d \geq 4$, $\beta = 1/2$

$$\sum_{i=\lfloor nt \rfloor - |\mathcal{R}^X_{\lfloor nt \rfloor}|+1}^{\lfloor nt \rfloor} 1_{\{U_i \leq i^{-1/2}\}} = \sum_{i=1}^{\lfloor nt \rfloor} 1_{\{U_i \leq i^{-1/2}\}} - \underbrace{\sum_{i=1}^{\lfloor nt \rfloor - |\mathcal{R}^X_{\lfloor nt \rfloor}|} 1_{\{U_i \leq i^{-1/2}\}}}_{:=|F_{\lfloor nt \rfloor}|} .$$

- We rewrite the process

$$B^n_t = \frac{1}{n^{1/2}} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i + \frac{|K_{\lfloor nt \rfloor}|}{n^{1/2}} \sum_{i \in K_{\lfloor nt \rfloor}} \frac{(\gamma_i - \xi_i)}{|K_{\lfloor nt \rfloor}|} .$$

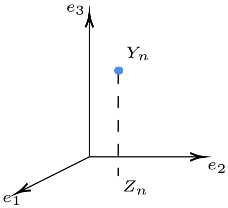
- We remember that

$$\frac{1}{n^{1/2}} \sum_{i=1}^{\lfloor nt \rfloor} 1_{\{U_i \leq i^{-1/2}\}} - \frac{|F_{\lfloor nt \rfloor}|}{n^{1/2}} \preceq \frac{|K_{\lfloor nt \rfloor}|}{n^{1/2}} \preceq \frac{|J_{\lfloor nt \rfloor}|}{n^{1/2}} .$$

Idea of the coupling

Let Y be a p_n-ERW in direction e_3 and Z is a lazy random walk in $\mathbb{Z}^2$.

For all $j \in \{1, 2\}$ and $i \geq 0$, $Y_i \cdot e_j = Z_i \cdot e_j$.



Lemma

If the process $\{Z_i\}_{i\geq 0}$ visits a new site then $\{Y_i\}_{i\geq 0}$ visits too.

As a direct consequence we obtain that $|\mathcal{R}_n^Y| \geq |\mathcal{R}_n^Z|$ for all $n \geq 0$.

Idea of the proof of convergence in distribution of the p_n-ERW, $d \geq 4$, $\beta = 1/2$

We set $|J_n'| := \sum_{i=1}^{\delta n} 1_{\{U_i \leq i^{-1/2}\}}$ where $\delta \in (\pi_d, 1]$ and obtain

$$\mathbb{P}[|J_n| \leq |J_n'|] = 1 \quad \text{as } n \to \infty\,.$$

By the coupling we have $|F_n| \leq \sum_{i=1}^{n-|\mathcal{R}_n^Z|} 1_{\{U_i \leq i^{-1/2}\}}$ for all $n \geq 1$, where Z is the lazy random walk define in the coupling in $\mathbb{Z}^{d-k}$.

We define $|F_n'| := \sum_{i=1}^{n-\delta' n} 1_{\{U_i \leq i^{-1/2}\}}$ where $\delta' \in (0, \pi_{d-k})$ and by Hamana and Kesten '01 we obtain

$$\mathbb{P}[|F_n| \leq |F_n'|] = 1 \quad \text{as } n \to \infty\,.$$

Idea of the proof of convergence in distribution of the p_n-ERW, $d \geq 4$, $\beta = 1/2$

- Thus as $n \to \infty$ we have

$$\mathbb{P}\left[\forall t \in [0,\infty) : \frac{\sum_{i=1}^{\lfloor nt \rfloor} 1_{\{U_i \leq i^{-1/2}\}}}{n^{1/2}} - \frac{|F'_{\lfloor nt \rfloor}|}{n^{1/2}} \leq \frac{|K_{\lfloor nt \rfloor}|}{n^{1/2}}\right] \to 1$$

$$\mathbb{P}\left[\forall t \in [0,\infty) : \frac{|K_{\lfloor nt \rfloor}|}{n^{1/2}} \leq \frac{|J'_{\lfloor nt \rfloor}|}{n^{1/2}}\right] \to 1 \quad \text{and consequently}$$

$$\mathbb{P}\left[\forall t \in [0,\infty) : 2t^{1/2}(1-(1-\delta')^{1/2}) \leq \frac{|K_{\lfloor nt \rfloor}|}{n^{1/2}} \leq 2(t\delta)^{1/2}\right] \to 1\,,$$

- Then for every limit point $Y_\cdot$ of a sub-sequence of $B^n_\cdot$, we obtain $W_t \cdot \ell_{D_k} + 2(1-(1-\delta')^{\frac{1}{2}})\mu_\gamma t^{\frac{1}{2}} \preceq Y_t \cdot \ell_{D_k} \preceq W_t \cdot \ell_{D_k} + 2\mu_\gamma \delta^{\frac{1}{2}} t^{\frac{1}{2}}$.

Conjecture about the range

Conjecture

Let X be a p_n-ERW in direction $\ell \in \mathbb{S}^{d-1}$, in $\mathbb{Z}^d$ with $d \geq 2$, $p_n = \mathcal{C}n^{-\beta} \wedge 1$, with $\beta \geq 1/2$. Then we have

$$\frac{|\mathcal{R}_n^X|}{n} \to \pi_d \quad as\ n \to \infty\ a.s..$$

where π_d is corresponding to $\{\xi_i\}_{i\geq 0}$.

Note that for $d = 2$, we have that $\pi_d = 0$, whereas for $d \geq 3$, $\pi_d \in (0, 1]$.

Idea of the proof that $\mathbb{P}[\eta(X_0) = \infty] > 0$

Let X be a p_n-GERW in direction ℓ, $d \geq 2$ and with $\beta < 1/6$. Then

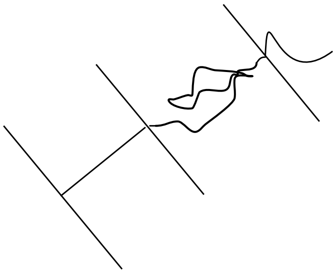
$$\mathbb{P}[\eta(X_0) = \infty] \geq \psi > 0\,.$$

Idea of the proof that $\mathbb{P}[\eta(X_0) = \infty] > 0$

- We prove the range of the p_n-GERW is large enough
- Under certain conditions we have that the p_n-GERW in ℓ direction with high probability

$$\mathbb{P}\left[X_n \cdot \ell < \frac{1}{3}\lambda n^{\frac{1}{2}+\alpha-\beta}\right] < 6n\exp(-\vartheta_1 n^{\vartheta_2})\,.$$

- Then using the uniformly elliptic condition



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Probability Theory and Related Fields

Idea of the proof of convergence in distribution of the p_n-ERW, $d=2$, $\beta=1/2$

- For any $\varepsilon>0$, we set the event $G:=\{|J_{\lfloor nT\rfloor}|>\varepsilon\sqrt{n}\}$. Then for a $\delta>0$, we have that

$$\mathbb{P}\left[\sup_{0\le t\le T}|J_{\lfloor nt\rfloor}|>\varepsilon\sqrt{n}\right]=$$
$$=\mathbb{P}[G\cap\{|\mathcal{R}^X_{\lfloor nT\rfloor}|>\delta\lfloor nT\rfloor\}]+\mathbb{P}[G\cap\{|\mathcal{R}^X_{\lfloor nT\rfloor}|\le\delta\lfloor nT\rfloor\}]$$
$$\le\mathbb{P}[|\mathcal{R}^X_{\lfloor nT\rfloor}|>\delta\lfloor nT\rfloor]+\frac{1}{\varepsilon\sqrt{n}}\sum_{i=1}^{\lceil\delta nT\rceil}\frac{1}{i^{1/2}}\,.$$

- Hence we obtain that $\limsup_{n\to\infty}\mathbb{P}\left[\sup_{0\le t\le T}|J_{\lfloor nt\rfloor}|>\varepsilon\sqrt{n}\right]\le c'(\delta T)^{\frac{1}{2}}/\varepsilon$.

Idea of the proof of the upper bound to the range of p_n-ERW in $d \geq 2$ and $\beta = 1/2$

Denote $\{N_i\}_{i\geq 0}$ as the sequence of times such that $N_i = \inf\{k > N_{i-1} : Z_k = 1\}$, where $Z_k \sim Bern(k^{-\frac{1}{2}})$ and $M_n = \inf\left\{i \geq 1 : \sum_{j=1}^{i} \Delta N_j \geq n\right\}$.

Let $\varepsilon \in (0,1)$ and we define $N_0 \equiv n^{\varepsilon}$.

$$|\mathcal{R}_n^X| \leq n^{\varepsilon} + M_n + \sum_{j=1}^{M_n} |\mathcal{R}^Y_{[N_{j-1}+2, N_j]}|,$$

where Y is a random walk whose increments are defined by $\{\xi_i\}_{i\geq 1}$.

Fix any $k \in \{1, 2, \dots, n\}$ and define $A_{n,k} := \{j \in \{1, 2, \dots, M_n\} : \Delta N_j \leq k\}$, clear $M_n \leq |A_{n,k}| + n/k$. Then

$$|\mathcal{R}_n^X| \leq n^{\varepsilon} + \frac{n}{k} + k|A_{n,k}| + \sum_{j\in A^c_{n,k}} |\mathcal{R}^Y_{[N_{j-1}+2, N_j]}|.$$

Idea of the proof that $\mathbb{P}[\eta(X_0)=\infty]>0$

For a sufficiently large integer m, consider the event

$$U_0 = \left\{(X_{k+1}-X_k)\cdot \ell \geq r \text{ , for all } \quad k=0,1,...,\lceil r^{-1}\rceil m-1\right\}.$$

We denote the time translation of $X: W_k = X_{\lceil r^{-1}\rceil m+k}$, $k \geq 0$. Then W is a p_n-GERW with $A' = \mathbb{Z}^d/\{X_0,\dots,X_{\lceil r^{-1}\rceil m-1}\}$ starting at $W_0 = X_{\lceil r^{-1}\rceil m}$ and for some k, we have $p_k = (q_0+\lceil r^{-1}\rceil m+k)^{-\beta}$.

Now we set $m = (C/\lceil r^{-1}\rceil)(3/\lambda)^{\frac{1}{\delta-1}}$ where $\delta=(2-\theta)(1/2+\theta)>1$, $\theta=\alpha-\beta$ and C is a positive constant, depending on α, β, q_0, K, λ and r.

Idea of the proof that $\mathbb{P}[\eta(X_0) = \infty] > 0$

For every $k \geq 1$ consider the following events

$$G_k = \left\{ \min_{\lfloor m_{k-1}^{2-\theta} \rfloor < j \leq m_k^{2-\theta}} \left(W_j - W_{\lfloor m_{k-1}^{2-\theta} \rfloor} \right) \cdot \ell > -m_k \right\} ,$$

$$U_k = \left\{ W_{\lfloor m_k^{2-\theta} \rfloor} \cdot \ell \geq m_{k+1} \right\} .$$

Where we denote $m_0 = 0$, $m_1 = m$ and, for $k \geq 1$, $m_{k+1} = \frac{1}{3} \lambda m_k^{\delta}$.

We show that

$$\left\{ X_n \cdot \ell > 0, \text{ for all } n \geq 1 \right\} \supset \left(\bigcap_{k=1}^{\infty} (G_k \cap U_k) \right) \cap U_0 .$$

Since $\mathbb{P}\left[\left(\cap_{k=1}^{\infty} (G_k \cap U_k) \right) \cap U_0 \right] = \mathbb{P}[U_0] \left(1 - \sum_{k=1}^{\infty} \mathbb{P}[G_k^c | U_0] + \mathbb{P}[U_k^c | U_0] \right)$

Idea of the proof that $\mathbb{P}[\eta(X_0) = \infty] > 0$

Controlling the probabilities:

- By the uniform ellipse condition,

$$\mathbb{P}\left[U_0\right] \geq h^{\lceil r^{-1} \rceil m} \, .$$

- By union bounds and Azuma's inequality:

$$\mathbb{P}[G_k|U_0] \geq 1 - m_k^{2-\theta} e^{-\frac{m_k^{\theta}}{2K^2}} \, .$$

- By Proposition 4:

$$\mathbb{P}[U_k|U_0] = \mathbb{P}\left[W_{\lfloor m_k^{2-\theta} \rfloor} \cdot \ell \geq \frac{\lambda}{3} m_k^{(2-\theta)(\frac{1}{2}+\theta)}\right] \geq 1 - 6e^{-\vartheta_1 m_k^{(2-\theta)\vartheta_2}} .$$